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Instability threshold in the Bénard-Marangoni problem

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Threshold values for the onset of Bénard-Marangoni convection and the critical dimensionless wave number which characterizes the size of the cells in a liquid layer heated from below are obtained by using linear perturbation techniques. The regions of instability in the three-dimensional space of the Marangoni, Rayleigh, and Biot numbers are determined for the coupled thermocapillary-buoyancy instability problem.

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The convective instability and the spontaneous formation of cell patterns in a fluid are important problems in physiochemical hydrodynamics, heat and mass transport, and the theory of self-organization in open systems. Bénard-Marangoni convection, named after Bénard's experiments [1,2] and the Marangoni effect [3,4], is one of the best known examples of these phenomena. To explain the origin of Bénard cells in a horizontal layer of fluid heated from below, Lord Rayleigh [5] developed his classical stability analysis of convection flow correlating the nature of pattern formation with the buoyancy effect. Block [6] and Pearson [7] proposed another mechanism of the formation of Bénard cells in a thin liquid film. According to their approach, the driving force of the convection is not buoyancy, but rather temperature-induced gradients in surface tension, i.e., the thermocapillary Marangoni effect. We know now that in the general case of arbitrary film thickness the mechanism of convection is mixed and involves the coupled thermocapillary-buoyancy instability. The surface-tension-drive effect (Marangoni instability) is dominant in thin layers less than 1 mm thick and the buoyancy effect (Rayleigh instability) predominates in layers about 1 cm and more.

In our work physical results are presented for the sufficient conditions for the onset of thermocapillary and coupled thermocapillary-buoyancy convection in a horizontal liquid layer with a nondeformable interface. Comparison with Pearson results are made for the "conducting" case [7] when the lower surface is a fixed-temperature boundary. We reveal the nonmonotonic dependence of the Marangoni number on the Biot number which has an extremum and is different from the correlation between the basic dimensionless parameters in Pearson's model.

The problem to be analyzed involves a horizontal liquid layer, of thickness h, heated from below and open to an ambient gas. The layer is assumed to be of infinite extent in the directions of the x and y axes, its upper free surface ($z=h$) is nondeformable whereas the bottom boundary is rigid ($z=0$). All physical properties of the liquid and the gas, except for surface tension, are assumed to be constant. The temperature at the bottom has a constant value T_0, and the heat balance at the upper surface ($z=0$) is described by Newton's law

$$-\kappa \partial T/\partial z = r(T-T_1) \tag{1}$$

where T is the temperature, T_1 is the temperature in the bulk of the gas phase for which the heat flux to the upper environment vanishes, κ is the thermal diffusivity of the liquid, and r is the heat transfer coefficient characterizing the heat removal from the free surface to the upper environment.

For the unperturbed motionless state the velocity vector $\mathbf{v}=0$, and the temperature distribution has a linear profile according to Fourier's law

$$\bar{T}=T_0-\beta z, \quad \beta=r(T_0-T_1)/(\kappa+rh). \tag{2}$$

Applying linear perturbation techniques to the equations of fluid motion and convective heat conduction we reduce them to

$$\nu\nabla^4 u-\partial\nabla^2 u/\partial t=0, \tag{3}$$

$$\kappa\nabla^2 T'-\partial T'/\partial t+\beta u=0, \tag{4}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, u is the velocity in the z direction, $T'=T-\bar{T}$ is the temperature perturbation, t is the time, and ν is the kinematic viscosity of the fluid.

The boundary conditions at upper and lower boundaries are

$$u=0, \quad \rho\nu\partial^2 u/\partial z^2=\alpha\nabla_1^2 T' \text{ at } z=h, \tag{5}$$

$$u=\partial u/\partial z=0, \quad T'=0 \text{ at } z=0, \tag{6}$$

where

$$\alpha=-(\partial\sigma/\partial T)_{\bar{T}(z=h)},$$

$\nabla_1^2=\partial^2/\partial x^2+\partial^2/\partial y^2$, ρ is the density of the fluid, σ is the surface tension, and α is the constant variation of surface tension with respect to temperature. The relation (5) allows for the thermocapillary Marangoni effect and follows from the equality of the tangential component of the viscous stress tensor and the stress due to the surface tension gradient. The effect of surface tension on the normal stress condition is neglected.

As new dimensionless variables we choose $x^*=x/h$, $y^*=y/h$, $z^*=z/h$, and $t^*=t\kappa/h^2$ (further omitting the asterisks), and we also introduce two dimensionless groups

$$\mathrm{Ma}=\frac{\alpha(T_0-T_1)h}{\rho\nu\kappa} \quad \text{(Marangoni number)},$$

$$\mathrm{Bi}=\frac{rh}{\kappa} \quad \text{(Biot number)}. \tag{7}$$

The Marangoni number Ma characterizes the ratio of the force induced by the surface tension gradient to the force related to viscous dissipation, and the Biot number Bi characterizes the ratio of the rate of heat removal from the interface to the environment to the rate of heat supply to the interface from the bulk of a liquid due to thermal conduction.

We consider an arbitrary disturbance and present the small perturbation u and T' in the following forms:

$$u=-(\kappa/h)U(z)\exp(ik_x x+ik_y y+\omega t), \tag{8}$$

$$T'=\beta h\Theta(z)\exp(ik_x x+ik_y y+\omega t), \tag{9}$$

where $k=(k_x^2+k_y^2)^{1/2}$ is the dimensionless wave number of the perturbations, and ω is the complex growth rate. Inserting Eqs. (3) and (4), we obtain a set of two ordinary differential equations, from which we find the functions $U(z)$ and $\Theta(z)$ containing six constants. To determine these constants, we use the boundary conditions in (1), (5), and (6) and obtain a uniform set of six algebraic equations relative to six unknowns having a nontrivial solution provided its characteristic determinant equals zero. From the characteristic equation for the eigenvalue problem we find the relation between the parameters Ma, Bi, k, and ω.

For the case of marginal stability ($\omega=0$) we obtain the relation

$$\mathrm{Ma}=\frac{4k(\mathrm{Bi}+1)(\mathrm{Bi}\sinh k+k\cosh k)(\sinh 2k-2k)}{\mathrm{Bi}(\sinh^3 k-k^3\cosh k)}. \tag{10}$$

Equation (10) allows one to investigate the dependence of the Marangoni number on the wave number for different values of the Biot number and to plot neutral stability curves, which separate the stability region (real part of $\omega<0$) under the curve and the instability region (real part of $\omega>0$) above the curve. We fix the values of Bi and then minimize Ma as a function of k to obtain the critical Marangoni number $\mathrm{Ma}_{\mathrm{cr}}$. All neutral stability curves have a minimum for any given value of the Biot number. The global minimum $\mathrm{Ma}_*=\min_{\mathrm{Bi}}(\min_k \mathrm{Ma})$ determines the threshold values for the onset of instability and the critical wave number

$$\mathrm{Ma}_*=222.54, \quad k_*=2.33 \text{ for } \mathrm{Bi}=1.54. \tag{11}$$

Figure 1 illustrates the correlation between the critical Marangoni number and the Biot number. The extremal character of this locus, which has $\mathrm{Ma}=\mathrm{Ma}_*$, can be attributed to the following physical reasons.

Increase in Bi from 0 to ∞ means a change in the thermal condition at the upper surface from "adiabatic" to "conducting." For a fixed temperature of the bottom a decrease in the Biot number and accordingly in the rate of heat removal from the upper surface to the environment leads to a decrease in the vertical thermal gradient and in the heat flux across the liquid layer. Therefore, the surface activity α and the Marangoni number, which are required for perturbation growth, have to increase by approaching the adiabatic condition at the free surface. In the limit, as $\mathrm{Bi}\to 0$, the temperature in the layer is equalized in the z direction, and the steady state $\bar{T}=T_0$ becomes stable and $\mathrm{Ma}_{\mathrm{cr}}\to\infty$.

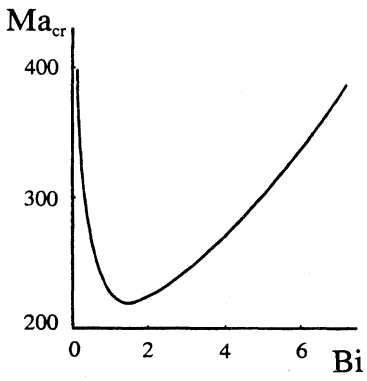

FIG. 1. The critical Marangoni number versus the Biot number for thermocapillary convection.

For the opposite case of a large Biot number, when the free surface is a good heat conductor, all temperature variations along the surface attenuate fast and the local tangential stresses related to surface tension are, therefore, small. Thus, in the limit $\mathrm{Bi}\to\infty$ the disturbance temperature gradients, which are the driving force for surface-tension-driven instability, vanish and no Marangoni convection can arise.

It is well known that the thermocapillary effect plays a dominant role in the development of natural convection in a fluid under microgravity and also on Earth in a thin liquid film. For layers more than 1 mm thick it is required to take into account both the motion caused by temperature-induced surface tension gradients and the buoyancy-driven convection flow in the gravitational field.

To consider the coupled thermocapillary-buoyancy instability problem, we use the Oberbeck-Boussinesq approximation [8] and then substitute the equation of motion involving the linear dependence of the density on the temperature for Eq. (3),

$$\nu\nabla^4 u-\partial\nabla^2 u/\partial t+g\gamma\nabla_1^2 T'=0, \tag{12}$$

where g is the gravitational acceleration, and γ is the coefficient of volume expansion.

We introduce the Rayleigh number as

$$\mathrm{Ra}=\frac{g\gamma(T_0-T_1)h^3}{\nu\kappa} \tag{13}$$

and analyze the linear stability problem (1), (2), (4), (5), (6), and (12). Following a calculation scheme similar to that in Ref. [9], we obtain the eigenvalue equation for the case of the marginal stability ($\omega=0$)

$$\det(A_{ij})=0, \quad i,\ j=1,2, \tag{14}$$

$$A_{11}=\sum_{n=1}^{\infty} aml, \quad A_{21}=\sum_{n=1}^{\infty}(-1)^n am,$$

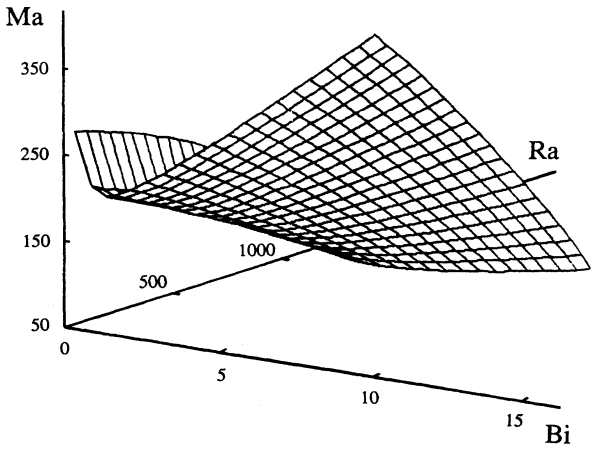

FIG. 2. Neutral stability surface in the space of Marangoni, Rayleigh, and Biot numbers for coupled thermocapillary-buoyancy convection.

$$A_{12}=bk^2\mathrm{Ma}\sum_{n=1}^{\infty}(-1)^n am(l+1),$$

$$A_{22}=-\tfrac{1}{2}(\mathrm{Bi}+1)+k^2\mathrm{Ma}\sum_{n=1}^{\infty}a[b(m+\mathrm{Ra})-l^2],$$

$$a=(l^3-bk^2\mathrm{Ra})^{-1},\quad b=\mathrm{Bi}/(\mathrm{Bi}+1),\quad l=(\pi n)^2+k^2,$$

$$m=(\pi n)^2.$$

Equation (14) involves the relationship between the basic dimensionless parameters—the Marangoni, Rayleigh, and Biot numbers—and the dimensionless wave number k. Using the described procedure of minimization with respect to k we obtain the dependencies of the critical Marangoni number $\mathrm{Ma}_{cr}=\min_k \mathrm{Ma}(k)$ and critical Rayleigh number $\mathrm{Ra}_{cr}=\min_k \mathrm{Ra}(k)$ on the Biot number. The results of the numerical solution of the neutral stability state equation (14) are shown in Fig. 2 as the locus in the three-dimensional space (Ma, Ra, Bi). The most interesting feature of Fig. 2 is that two-dimensional boundaries between stability and instability regions $\mathrm{Ma}_{cr}(\mathrm{Bi})$ and $\mathrm{Ra}_{cr}(\mathrm{Bi})$ have local minima. We find that $\mathrm{Ma}_{cr}(\mathrm{Bi})\to\mathrm{Ma}_*\approx 222$ if $\mathrm{Ra}\to 0$, $\mathrm{Bi}\to 1.54$ and $\mathrm{Ra}_{cr}(\mathrm{Bi})\to\mathrm{Ra}_*\approx 1100$ if $\mathrm{Ma}\to 0$, $\mathrm{Bi}\to\infty$. Here Ma_* is the instability threshold for pure thermocapillary convection and Ra_* is the well-known critical value for Rayleigh convection in a liquid layer with rigid-free boundaries [10].

Let us compare our results with the data in the pioneering work of Pearson [7], who first suggested the model for surface-tension-driven instability in a horizontal layer of a fluid heated from below. This work initiated most of the subsequent investigations on thermocapillary convection. The basic dimensionless parameters introduced by Pearson are $B=\alpha\beta h^2/\rho\nu\kappa$ and $L=qh/\lambda$ where β is the temperature gradient for the unperturbed state, q is the rate of change with the temperature of the time rate of heat loss per unit area from the upper surface, and λ is the coefficient of heat conduction in the liquid. Note that the parameter L is equal to the Biot number introduced above. Two independent parameters T_0 and β determine the unperturbed temperature profile for the "conducting" case in Ref. [7], p. 491, but the conditions for the onset of thermocapillary convection depend only on the parameter β. Pearson used the different boundary conditions at the free surface ($z=0$): $-\partial T/\partial z=\beta=\mathrm{const}$ is for the unperturbed state and $-\lambda\partial T/\partial z=qT$ is for the perturbed state. As a result, it was obtained that the correlation between B and L has a monotonously increasing character and is close to linear. The limit $L=0$ or $\mathrm{Bi}=0$ corresponds to zero temperature gradient, $\beta\to 0$ if $L\to 0$. In this case, as was described above, the thermocapillary convection cannot originate and the instability threshold, as physical value, should go into infinity, that is, $\mathrm{Ma}_{cr}\to\infty$ if $\mathrm{Bi}\to 0$. On the other hand, the instability threshold in [7] corresponded to the finite critical value $B_{cr}\approx 80$ if $L=0$. If the boundary condition (1) had been applied to both the unperturbed and the perturbed state within Pearson's model, the parameter β would have been unambiguously determined by parameters T_0, κ, λ, q, and h and would have been equal to $\beta=(T_0/h)L/(L+1)$. Therefore, the parameter B, *a priori*, would be a function of the parameter L and they should not be considered as independent numbers. Due to the above reasons, instead of B, it is preferable to introduce the Marangoni number according to (7) in the case of a fixed temperature at the lower surface. This case corresponds to the conditions for numerous experiments on Bénard convection [10,11].

Finally, note that the results obtained here and the physical conclusions drawn are easily transferable to solutocapillary convection [12] which takes place in mass transport due to diffusion and when the surface tension depends on the concentration of the solute.

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